A tree-based scaling exponent for random cluster models

1992 J. Phys. A: Math. Gen. 25 L1065

(http://iopscience.iop.org/0305-4470/25/17/009)

J. Phys. A: Math. Gen. 25 (1992) L1065–L1069. Printed in the UK

LETTER TO THE EDITOR

A tree-based scaling exponent for random cluster models

David Aldous and Bret Larget
Department of Statistics, University of California, Berkeley, CA 94720, USA

Received 6 April 1992, in final form 23 June 1992

Abstract. For models of random trees in the hypercubic lattice, or more general random clusters to which tree structure may be added, we introduce a novel scaling exponent. For the particular model of uniform random spanning trees in two and three dimensions, its value is estimated by Monte Carlo simulation.

There has been much study of models of random connected clusters of sites in the hypercubic lattice, both dynamic growth models such as diffusion-limited aggregation [19, 10, 17] and combinatorial models such as lattice trees and animals [9, 11–13], other polymer models [15], and self-avoiding walks [5]. Typical questions asked of such models are:

(i) fractal dimension D_d of the cluster in low dimensions d;

(ii) critical dimension d_0 above which the predictions of the mean-field theory are correct.

We introduce a new methodology which focuses more on the way a cluster is connected than on its shape. (As observed by a referee, different notions of connectivity have been studied [18, 6–8, 16] in the context of percolation problems.) Our methodology is directly applicable to models of trees, and may be indirectly applied to more general cluster models by incorporating tree structure. Thus in a growth model, when the nth particle p_n is attached it is the neighbour of one or more existing particles, and we may include in the model description a way to choose one of those neighbours p (uniformly, if a more natural method is not available) and add an edge (p, p_n) to the existing tree. We regard trees as rooted at the origin. Now the intrinsic structure of an n-vertex tree may be represented by a *walk* of length $2n$ with ± 1 steps and with first return to 0 at step $2n$. Order the children of each vertex v as first, second, third, etc (in a dynamic growth model, use order of attachment; in a combinatorial model, order randomly). Consider *depth-first search* of the tree, moving from a current vertex v to the first child not previously visited, if one exists, and otherwise time $2n - 1$. Define the associated walk $1 = w(1), w(2), \ldots, w(2n-1) = 1$ to take a $+1$ step when the search moves from parent to child and a -1 step when the search moves from child to parent (by convention $w(0) = w(2n) = 0$); see figure 1. Note that the value of w when the search is at vertex v is 1 plus the within-tree distance (i.e. length of unique path in the tree) from v to the origin.

This transformation from tree to walk captures the intrinsic graph-theoretic structure of the tree, but not the way the tree was embedded into the lattice, which is the major disadvantage of the method. Its advantage is that it enables us to transform

0305-4470/92/171065+05$04.50

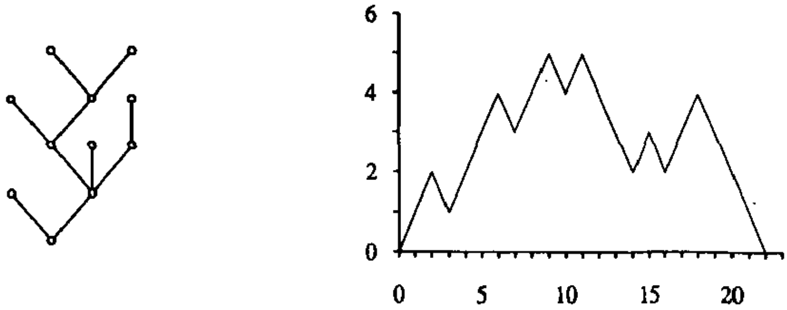

Figure 1. A small abstract tree and its associated walk.

study of a random cluster to study of a random walk (in general, with some complicated probability distribution). This is an advantage because studying convergence of rescaled discrete random walks to continuous random functions is simpler (as mathematical theory, and in seeking to observe such convergence in simulations) than directly studying convergence of rescaled random subsets of the lattice to random subsets of the continuum. More specifically, it enables one to compare simulation behaviour of random clusters or trees in the lattice with the *rigorous* limit theory [2,3] for abstract (i.e. non-lattice) random trees. For natural combinatorial models of random n-vertex abstract trees (e.g. unordered labelled, ordered or ordered binary) it is known that

$$n^{-1/2}w(2nt) \xrightarrow{d} 2aW(t) \qquad \text{as} \quad n \to \infty \tag{1}$$

where $\xrightarrow{d}$ is convergence in distribution [4] and where W is Brownian excursion, i.e. one-dimensional Brownian motion conditioned on $W(0) = W(1) = 0$, $W(t) > 0$ for $0 < t < 1$. Thus we rescale each ± 1 step to a $\pm n^{-1/2}$ step and rescale the discrete time interval $[0, 2n]$ to the interval $[0, 1]$. In (1) a is a scale parameter depending on the exact model. For models of random lattice clusters for which there is a natural way to incorporate tree structure, one may ask whether there is a critical dimension d_0 such that (1) is true for $d \geqslant d_0$ but not for $d < d_0$, and study this question by Monte Carlo simulation. The intuitive interpretation of the existence of such a critical dimension d_0 is that the geometry of d-dimensional space has a major influence on cluster behaviour for $d < d_0$ but not for $d \geqslant d_0$. More generally one may ask whether in a fixed dimension the model admits a limit

$$n^{-\alpha}w(2nt) \xrightarrow{d} W^*(t) \qquad \text{as} \quad n \to \infty \tag{2}$$

for some limit process W^*. When (2) holds, α is our novel *scaling exponent*. Such a result would be an elaboration of the idea that the mean within-tree distance between vertices of an n-site cluster grows as order n^α. Note that for trees grown in a regular deterministic manner (cf figure 2) the walks typically oscillate increasingly rapidly as $n \to \infty$ and therefore can have no limit process (2).

In this letter we study one model, the uniform random spanning tree. Consider the discrete torus Z_N^d with $n = N^d$ vertices, i.e. the hypercube of side-length N continued periodically. This graph has a finite set of spanning trees, so it makes sense to talk of a uniform random spanning tree, and there is an efficient algorithm (see

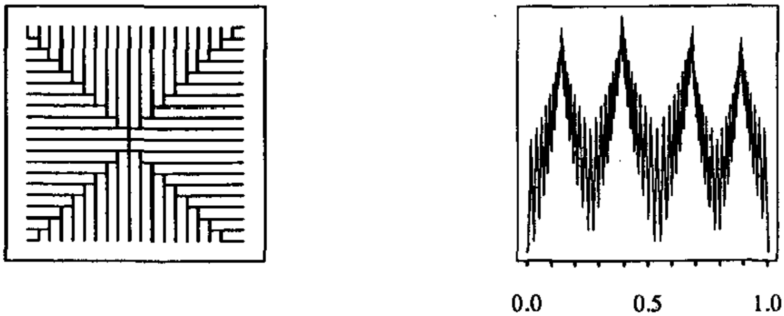

Figure 2. A spanning tree in two dimensions where every vertex is the minimum possible distance from the root and its associated walk.

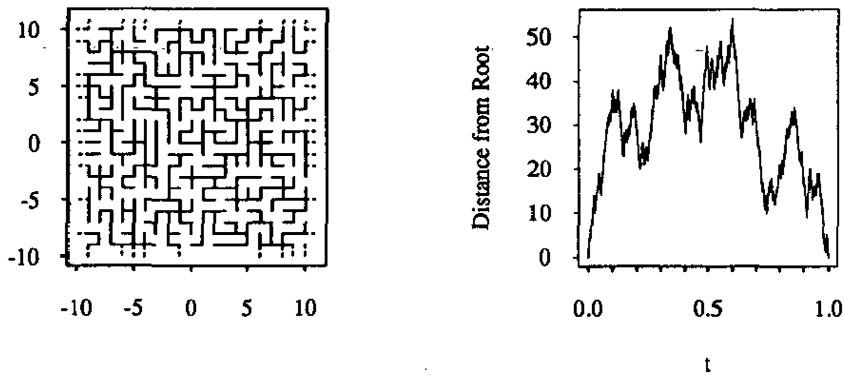

Figure 3. A randomly determined spanning tree in two dimensions and its associated walk. The broken lines indicate edges which connect two vertices on opposite sides.

e.g. [1]) for Monte Carlo simulation of spanning trees in any finite graph. Figure 3 is a realization of a spanning tree ($d = 2$, $N = 20$) and its associated walk. There are theoretical reasons to believe that for large d the Brownian excursion approximation (1) holds, and hence $\alpha = \frac{1}{2}$. But (1) is implausible for $d = 2$ because it would imply that the mean within-tree distance from a vertex to the origin is order $n^{1/2} = N$, the same order as the minimum possible distance.

This problem can be studied analytically with $d = 1$. In this case, (2) holds with $\alpha = 1$ and $W^*(t)$ defined as follows. Choose u uniformly from the unit interval. Let $W^*(0) = W^*(u) = W^*(1) = 0$. Let $W^*(u/2) = u$ and $W^*((u+1)/2) = 1 - u$. Then $W^*(t)$ is the piecewise-linear function connecting these five points.

We studied simulations with $N = 40$, 80 and 160 ($d = 2$) and with $N = 20$, 30 and 40 ($d = 3$), using 2000 repetitions at each size.

Let $\bar{w}_N = (2N^d+1)^{-1}\sum_{k=0}^{2N^d} w(k)$ be the mean height (within-tree distance from root plus 1) over all steps in the associated walk. Then (2) predicts $\bar{w}_N \sim a_d N^{d\alpha}$. Plotting $\log \bar{w}_N$ against $\log N$ gives the estimates

$$\alpha \approx 0.628 \quad (d = 2) \qquad \alpha \approx 0.544 \quad (d = 3). \tag{3}$$

To elaborate this analysis, let $\bar{w}_N(t)$ be the mean of $w(2nt)$. Then (2) predicts

$$\bar{w}_N(t) \approx N^{d\alpha}\bar{w}_\infty(t) \tag{4}$$

where $\bar{w}_\infty(t) = EW^*(t)$. Figures 4 and 5 show the curves $N^{-d\alpha}\bar{w}_N(t)$ with α given by (3). The curves stay very close for varying N, consistent with (4). For Brownian excursion it is known that $EW(t) = c\sqrt{t(1-t)}$. For a uniform random spanning tree with $d = 1$, $EW^*(t) = 2t(1-t)$. It is not hard to argue that, given (2), the limit should scale like $\bar{w}_\infty(t) \approx t^\alpha$ as $t \downarrow 0$. So one might guess

$$\bar{w}_N(t) \approx c_d N^{d\alpha}\ [t(1-t)]^\alpha$$

and indeed the curves in figures 4 and 5 are a good fit to $c_d[t(1-t)]^\alpha$.

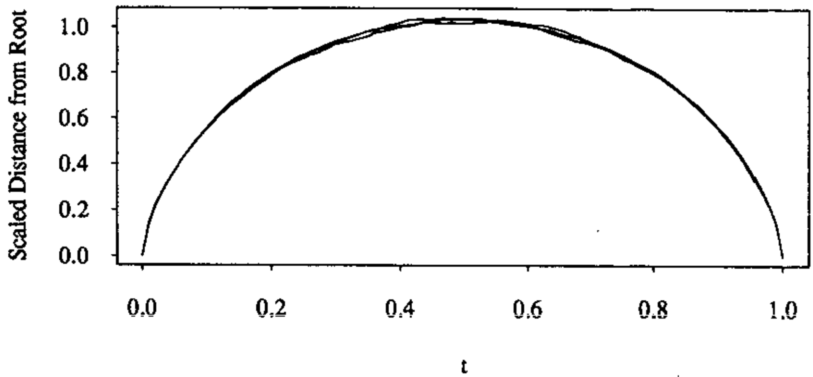

Figure 4. Mean scaled distance from root ($d = 2$) for $N = 40$, 80 and 160.

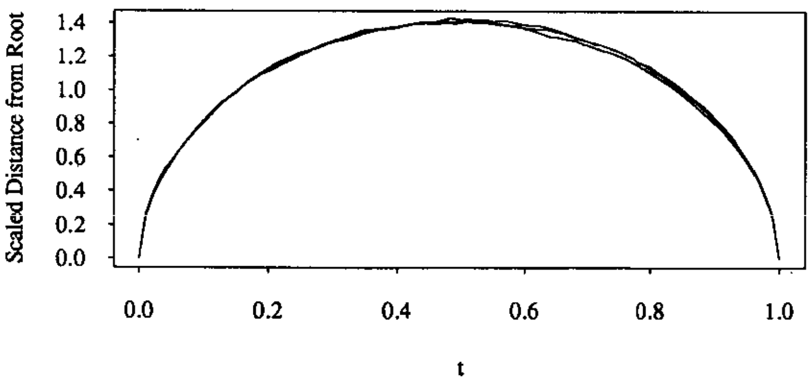

Figure 5. Mean scaled distance from root ($d = 3$) for $N = 20$, 30 and 40.

From the numbers in (3) one guesses that in $d = 4$, α is either exactly $\frac{1}{2}$ or very close to $\frac{1}{2}$, and it seems impractical to distinguish these possibilities by simulation. Let us remark that for a different aspect of this model (is the $N \to \infty$ limit a forest of disjoint trees or a single tree?) it is rigorously known [14] that the critical dimension is 5. In view of the connection [2, 14] between random walk and spanning trees it would be interesting to know whether the scaling exponents (3) relate to any scaling exponents previously studied for a d-dimensional random walk. Finally, we record a puzzling observation from the simulations. Table 1 shows for $d = 3$ the Monte

Table 1. Monte Carlo estimates of the frequency $p(i)$ of vertices with out-degree i for $d = 3$, compared with the predictions of equation (5).

Out-degree	0	1	2	3	4	5
Monte Carlo	0.3275	0.4099	0.2049	0.0510	0.0064	0.0003
Equation (5)	0.3279	0.4094	0.2047	0.0512	0.0064	0.0004

Carlo estimates of the frequency $p(i)$ of vertices with out-degree i. The data is a remarkable fit to the unique mean-1 distribution such that

$$p(i) = p(i-1)/2^i \qquad i \geqslant 2 \tag{5}$$

although we have no theoretical explanation for (5).

In conclusion, the particular model studied here was chosen for mathematical simplicity rather than physical relevance. A possible use of this methodology more relevant to physicists would be to consider cases where two different random cluster models appear in simulations to have similar fractal growth exponents according to the usual ways of estimating fractal dimension, and study whether the tree-based scaling exponents are similar. This seems a potentially powerful method for supporting or rejecting the belief that the different models lead to similar observed behaviour.

David Aldous' research was supported by NSF grant DMS90-01710.

References

[1] Aldous D 1990 *SIAM J. Discrete Math.* 3 450
[2] Aldous D 1991 *Stochastic Analysis* ed M T Barlow and N H Bingham (Cambridge: Cambridge University Press) pp 23–70
[3] Aldous D 1992 The continuum random tree III *Ann. Probab.* submitted
[4] Billingsley 1968 *Convergence of Probability Measures* (New York: Wiley)
[5] Brydges D and Spencer T 1985 *Commun. Math. Phys.* **97** 125
[6] Cherry R and Domb C 1980 *J. Phys. A: Math. Gen.* **13** 1325
[7] Domb C 1974 *J. Phys. C: Solid State Phys.* **7** 2677
[8] Domb C, Schneider T and Stoll E 1975 *J. Phys. A: Math. Gen.* **8** L90
[9] Gaunt D S, Sykes M F, Torrie G M and Whittington S G 1982 *J. Phys. A: Math. Gen.* **15** 3209
[10] Halsey T C, Jensen M H, Kadanoff L P, Procaccia I and Shraiman B I 1986 *Phys. Rev.* A **33** 1141
[11] Hara Y and Slade G 1990 *J. Stat. Phys.* **59** 1469
[12] Lubensky T C and Isaacson J 1979 *Phys. Rev.* A **20** 2130
[13] Madras N, Soteros C E and Whittington S G 1988 *J. Phys. A: Math. Gen.* **21** 4617
[14] Pemantle R 1991 *Ann. Probab.* **19** 1559
[15] Privman V and Surakic N M 1989 *Directed Models of Polymers, Interfaces and Clusters: Scaling and Finite-size Properties* (Berlin: Springer)
[16] Stoll E and Domb C 1978 *J. Phys. A: Math. Gen.* **11** L57
[17] Vicsek T 1989 *Fractal Growth Phenomena* (Singapore: World Scientific)
[18] Whittington S G, Middlemiss K M and Gaunt D S 1981 *J. Phys. A: Math. Gen.* **14** 2415
[19] Witten T A and Saunder L M 1981 *Phys. Rev. Lett.* **47** 1400